

# Solutions of Podolsky's Electrodynamics Equation in the First-Order Formalism

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## Abstract

The Podolsky generalized electrodynamics with higher derivatives is formulated in the first-order formalism. The first-order relativistic wave equation in the 20-dimensional matrix form is derived. We prove that the matrices of the equation obey the Petiau-Duffin-Kemmer algebra. The Hermitianizing matrix and Lagrangian in the first-order formalism are given. The projection operators extracting solutions of field equations for states with definite energy-momentum and spin projections are obtained, and we find the density matrix for the massive state. The  $13 \times 13$ -matrix Schrodinger form of the equation is derived, and the Hamiltonian is obtained. Projection operators extracting the physical eigenvalues of the Hamiltonian are found.

## 1 Introduction

There is currently a renewal of interest in higher derivative (HD) field theories. HD field equations appear in many models such as renormalizable quantum gravity [1], Podolsky's generalized electrodynamics [2], the Lee-Wick model [3] and others. One of the reasons to consider HD theories is to improve renormalization properties of theories and to remove ultraviolet divergences [4]. However, HD models suffer some difficulties connected with the presence of ghosts. These can lead to the violation of unitarity [5], [6]. Nevertheless, in some HD models these problems with negative probabilities and S-matrix unitarity can be avoided [7]. Also, the quadratic divergence associated with the Higgs mass were removed in the HD Lee-Wick standard model [8], that solves the hierarchy problem. Extensions of the minimal standard model living to new physics are justified until observations at the Large Hadron Collider (LHC) will be analyzed.

It is well known that in classical electrodynamics the electromagnetic mass is infinite and, therefore, there are infinities associated with a point

particle. One of the ways to solve this problem in classical theory is to use the Lorentz invariant regularization of the Maxwell equations at short distances. With the help of an appropriate cutoff the point particle limit can be achieved. This programme was realized in Podolsky's electrodynamics. Firstly the interest to Podolsky's electrodynamics was due to the finiteness of the theory: the electromagnetic energy of a point charge is finite contrarily to ordinary electrodynamics. If distances are much greater than a cutoff, Podolsky's electrodynamics converts into Maxwell's electrodynamics. The solution to Poisson equation for the potential corresponding to a point charge  $e$ , located at the origin, in Podolsky's electrostatics is given by [2]

$$\varphi = \frac{e}{4\pi r} \left(1 - e^{-r/a}\right),$$

where  $a$  is a new parameter of the theory with the dimension of the length playing the role of the cutoff. This potential becomes the Coulomb potential at distances much bigger than  $a$ . At  $r \rightarrow 0$  the potential  $\varphi$  approaches the finite value  $e/4\pi a$ . The energy of the field for a point charge is also finite in the hole space. Thus, the electrostatic energy can be considered as the regularized electromagnetic mass of a point charge. It was shown [9] that higher derivatives terms in Podolsky's equations suppress unphysical runaway solutions with exponentially growing acceleration of the Abraham-Lorentz equation. There are not unwanted solutions if the cutoff is greater than half of the electron classical radius. The upper bound on the parameter is of the order  $a \sim 10^{-16}$  cm [9], i.e. the same as the Compton wavelength of the neutral  $Z$ -boson. Classical Maxwell's electrodynamics is not valid at small distances and time intervals due to quantum effects. It was also mentioned in [10] that in the framework of non-relativistic quantum theory a natural cutoff of order of the electron Compton wavelength is effectively appeared by QED processes in close analogy with the classical theory of extended charges. Thus, one may treat the classical Podolsky's electrodynamics as an effective theory where a cutoff introduced,  $a$ , is due to the quantum processes at small distances (large momentum). If distances are larger than  $a$ , the classical regime begins.

At the same time although QED describes all experimental data well, there are some internal difficulties with the regularization [11]. We mention infrared catastrophe: when the average number of photons  $\bar{n} \rightarrow \infty$ , then the matrix element  $|\langle 0 \text{ out} | 0 \text{ in} \rangle| \rightarrow 0$ , and it is impossible to construct the "out" Fock space from the "in" space, nor to find unitary operator  $S$

[11]. The authors [12] wrote: "*There is an alternative possibility to avoid infrared divergences. We give the photon a small mass  $\mu$ . This will cut off the low-energy region since now  $k^0 > \mu$  and therefore remove the infrared divergence.... The infrared divergences of quantum electrodynamics are essentially classical*". We continue with the citation [13]: "*Another aspect of infrared singularities related to the long-range character of the Coulomb forces. The latter induces an infinite phase shift on the scattered plane waves. To prevent it we may introduce a screening factor which in a **consistent theory** would be related to the fictitious photon mass  $\mu$* ". Thus, in QED the cutoff is introduced "by hands" as for small distances (to remove ultraviolet divergences) as well as for large distances (to avoid infrared catastrophe). Therefore, one may consider naturally to extend classical Podolsky's electrodynamics on the quantum level where the cutoff is appeared due to the presence of higher derivatives. Anyway, different aspects of Podolsky's electrodynamics, in our opinion, have a definite theoretical interest.

Some features of Podolsky's electrodynamics were investigated in [14], [15], [16], [17]. The goal of this paper is to formulate Podolsky's electrodynamics equation in the form of the first-order relativistic wave equation, and to obtain solutions in the form of projection matrices.

The paper is organized as follows. In Sec. 2, the third-order field equation is discussed. We derive the first-order relativistic wave equation for Podolsky's electrodynamics in the 20-dimensional matrix form. The Hermitianizing matrix and the Lagrangian in the matrix form are found in Sec. 3. The projection operators extracting solutions of field equations for definite energy and spin states of particles are obtained in Sec. 4. We find the density matrix for the massive state. In Sec. 5 the  $13 \times 13$ -matrix Schrodinger form of the equation is derived, and the Hamiltonian is obtained. Solutions of this equation are found in the form of projection operators. The results are discussed in Sec. 6. In Appendix A, we consider the first-order wave equation in the presence of the charge current density. The Lorentz covariance of the equation is proven. Some useful products of matrices are derived in Appendix B. We obtain "minimal" polynomials of the matrix of the equation for massless and massive states. In Appendix C the "minimal" polynomial of the Hamiltonian matrix is derived.

The Heaviside's units are chosen,  $\hbar = c = 1$ , and Euclidian metric is used,  $x_\mu = (x_m, ix_0)$ . Greek letters range from 1 to 4 and Latin letters range from 1 to 3, and there is a summation on repeated indexes.

## 2 Field equations

### 2.1 Third-order field equations

The Lagrangian of Podolsky's electrodynamics is given by [2]

$$\mathcal{L}_P = -\frac{1}{2} \left[ \frac{1}{2} F_{\mu\nu}^2 + a^2 (\partial_\mu F_{\nu\mu})^2 \right], \quad (1)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength,  $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it))$ . The dimensional parameter  $a$  can be written as  $a = 1/m$ , where  $m$  is the mass parameter. The Euler-Lagrange equations follow from Eq.(1):

$$(\partial_\alpha^2 - m^2) \partial_\mu F_{\nu\mu} = 0. \quad (2)$$

The Lagrangian (1) and the equation of motion (1) are gage-invariant under the  $U(1)$ -group. We can represent Eq.(2), in the momentum space as the matrix equation:

$$(p^2 + m^2) (p^2 - p \cdot p) A = 0, \quad (3)$$

where  $A = \{A_\mu\}$ , the matrix-dyad  $p \cdot p$ , with matrix elements  $(p \cdot p)_{\mu\nu} = p_\mu p_\nu$ , is introduced, and the four-momentum being  $p_\mu = (\mathbf{p}, ip_0)$ . The matrix  $M = p^2 - p \cdot p$  obeys the minimal polynomial  $M(M - p^2) = 0$ , so that the eigenvalues of the matrix  $M$  are zero and  $p^2$ . Thus, Eq.(3) leads to the dispersion equation:

$$p^2 (p^2 + m^2) = 0. \quad (4)$$

Eq.(4) shows that there are massless and massive states in the spectrum. The propagator of fields is given by

$$\frac{m^2}{p^2 (p^2 + m^2)} = \frac{1}{p^2} - \frac{1}{p^2 + m^2}. \quad (5)$$

The first term in Eq.(5) is the propagator of the photon massless field and the second term corresponds to the propagator of the massive state of the field. A "wrong" sign  $(-)$  in Eq.(5) indicates that the massive field state is a ghost. As a result, the massive field state gives the negative contribution to the energy [2], and the classical Hamiltonian is unbounded. To have the positive eigenvalues of the Hamiltonian in the second quantized theory, one has to introduce the indefinite metric. The commutation relations for creation, annihilation operators of the massive state have the wrong sign  $(-)$

[2]. The Hilbert space of states is the direct sum of the two subspaces  $H_p$  and  $H_n$  with positive ( $H_p$ ) and negative ( $H_n$ ) square norms. The massless states correspond to a positive square norm, and massive states — to a negative square norm. The transitions between two subspaces  $H_p$  and  $H_n$  break the unitarity of the theory. But if the mass  $m \rightarrow \infty$  such transitions are forbidden and the unitarity is recovered. Thus, a ghost can be removed in the theory at large  $m$ . This procedure is similar to the Pauli-Villars regularization of Feynman diagrams. Therefore, there is physical sense of the Podolsky theory. We also can argue (similar to Lee-Wick model [8]) that there is no problem with unitarity if the massive photon decays to ordinary fermions through its couplings and is not in the spectrum.

## 2.2 First-order field equations

Now we reformulate the third-order field equation (2) in the form of first-order relativistic wave equation. Let us consider the system of first-order equations

$$\partial_\mu \psi_{\nu\mu} + m\tilde{\psi}_\nu = 0, \quad (6)$$

$$\partial_\nu \psi_\mu - \partial_\mu \psi_\nu + m\psi_{\mu\nu} = 0, \quad (7)$$

$$\partial_\mu \tilde{\psi}_{\nu\mu} + m\tilde{\psi}_\nu = 0, \quad (8)$$

$$\partial_\nu \tilde{\psi}_\mu - \partial_\mu \tilde{\psi}_\nu + m\tilde{\psi}_{\mu\nu} = 0, \quad (9)$$

where

$$\psi_\mu = mA_\mu, \quad \psi_{\mu\nu} = F_{\mu\nu}, \quad \tilde{\psi}_\mu = \frac{1}{m}\partial_\nu F_{\nu\mu}, \quad \tilde{\psi}_{\mu\nu} = \frac{1}{m^2}\partial_\alpha^2 F_{\mu\nu}. \quad (10)$$

After replacing  $\tilde{\psi}_\nu$  from Eq.(6) and  $\tilde{\psi}_{\mu\nu}$  from Eq.(9) into Eq.(8), one obtains Eq.(2). Eq.(7) is the usual equation for the potentials. Thus, we claim that the system of first-order equations (6)-(9) is equivalent to the third-order Eq.(2). Let us introduce the 20-dimensional wave function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi_\mu(x) \\ \psi_{\mu\nu}(x) \\ \tilde{\psi}_\mu(x) \\ \tilde{\psi}_{\mu\nu}(x) \end{pmatrix} \quad (A = \mu, [\mu\nu], \tilde{\mu}, [\widetilde{\mu\nu}]), \quad (11)$$

where  $\psi_{[\mu\nu]}(x) = \psi_{\mu\nu}(x)$ ,  $\psi_{\tilde{\mu}}(x) = \tilde{\psi}_{\mu}(x)$ ,  $\psi_{\widetilde{[\mu\nu]}}(x) = \tilde{\psi}_{\mu\nu}(x)$ . The function  $\Psi(x)$  represents the direct sum of two four-vectors  $\psi_{\mu}(x)$ ,  $\tilde{\psi}_{\mu}(x)$ , and two antisymmetric tensors of the second rank  $\psi_{\mu\nu}(x)$ ,  $\tilde{\psi}_{\mu\nu}(x)$ .

We explore the elements of the entire matrix algebra  $\varepsilon^{A,B}$  [18], [19] with matrix elements and products

$$\left(\varepsilon^{M,N}\right)_{AB} = \delta_{MA}\delta_{NB}, \quad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \quad (12)$$

where  $A, B, M, N = \mu, [\mu\nu], \tilde{\mu}, \widetilde{[\mu\nu]}$ , and generalized Kronecker symbols

$$\delta_{[\mu\nu][\alpha\beta]} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}.$$

The  $\varepsilon^{M,N}$  are  $20 \times 20$ -matrices, that consist of zeros and only one element is unity where the row  $M$  and the column  $N$  cross.

With the help of Eq.(11),(12) the system of equations (6)-(9) can be represented in the form of the first-order equation

$$\begin{aligned} & \partial_{\mu} \left( \varepsilon^{\nu, [\nu\mu]} + \varepsilon^{[\nu\mu], \nu} + \varepsilon^{\tilde{\nu}, \widetilde{[\nu\mu]}} + \varepsilon^{\widetilde{[\nu\mu]}, \tilde{\nu}} \right)_{AB} \Psi_B(x) \\ & + m \left( \frac{1}{2} \varepsilon^{[\nu\mu], [\nu\mu]} + \varepsilon^{\nu, \tilde{\nu}} + \varepsilon^{\tilde{\nu}, \tilde{\nu}} + \frac{1}{2} \varepsilon^{\widetilde{[\nu\mu]}, \widetilde{[\nu\mu]}} \right)_{AB} \Psi_B(x) = 0. \end{aligned} \quad (13)$$

There is a summation over all repeated indices. We define 20-dimensional matrices as follows:

$$\beta_{\mu} = \beta_{\mu}^{(1)} + \tilde{\beta}_{\mu}^{(1)}, \quad \beta_{\mu}^{(1)} = \varepsilon^{\nu, [\nu\mu]} + \varepsilon^{[\nu\mu], \nu}, \quad \tilde{\beta}_{\mu}^{(1)} = \varepsilon^{\tilde{\nu}, \widetilde{[\nu\mu]}} + \varepsilon^{\widetilde{[\nu\mu]}, \tilde{\nu}}, \quad (14)$$

$$P = \frac{1}{2} \varepsilon^{[\nu\mu], [\nu\mu]} + \varepsilon^{\nu, \tilde{\nu}} + \varepsilon^{\tilde{\nu}, \tilde{\nu}} + \frac{1}{2} \varepsilon^{\widetilde{[\nu\mu]}, \widetilde{[\nu\mu]}}. \quad (15)$$

Taking into account Eq.(14),(15), Eq.(13) takes the form of the first-order relativistic wave equation:

$$(\beta_{\mu} \partial_{\mu} + mP) \Psi(x) = 0. \quad (16)$$

The presence of the projection operator  $P$  in Eq.(16) is connected with the fact that there is a massless state in the spectrum [19], [20]. Thus, we reformulated the higher derivative equation (2) in the form of the first-order equation (16). The  $P$  is the projection operator,  $P^2 = P$  [21] and it is not the Hermitian matrix  $P^+ \neq P$ . The matrices  $\beta_{\mu}^{(1)}$  and  $\tilde{\beta}_{\mu}^{(1)}$  are Hermitian matrices and have non-zero components in 10-dimensional subspaces  $(\mu, [\mu\nu])$ ,

$(\tilde{\mu}, [\widetilde{\mu\nu}])$ , respectively and obey the Petiau-Duffin-Kemmer algebra [22], [23] (see also [18], [19]):

$$\beta_\mu\beta_\nu\beta_\alpha + \beta_\alpha\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\alpha + \delta_{\alpha\nu}\beta_\mu. \quad (17)$$

Therefore, the matrix  $\beta_\mu$  is the direct sum of two 10-dimensional Petiau-Duffin-Kemmer matrices. The projection operator  $P$  “connects” two 10-dimensional subspaces  $(\mu, [\mu\nu])$ , and  $(\tilde{\mu}, [\widetilde{\mu\nu}])$ . Thus, HD Podolsky’s electrodynamics equations lead to “doubling” the dimension of the Petiau-Duffin-Kemmer algebra representation.

### 3 The Lorentz covariance and Hermitianizing matrix

Let us prove the Lorentz covariance of Eq.(16). The Lorentz group transformations of coordinates are given by  $x'_\mu = L_{\mu\nu}x'_\nu$ , where the Lorentz matrix  $L = \{L_{\mu\nu}\}$  satisfies the equation  $L_{\mu\alpha}L_{\nu\alpha} = \delta_{\mu\nu}$ . The wave function (11), under the Lorentz coordinates transformations, becomes

$$\Psi'(x') = T\Psi(x), \quad (18)$$

where  $20 \times 20$ -matrix  $T$  realizes the reducible tensor representation of the Lorentz group. The first-order wave equation (16) is transformed into

$$(\beta_\mu\partial'_\mu + mP)\Psi'(x') = (\beta_\mu L_{\mu\nu}\partial_\nu + mP)T\Psi(x) = 0, \quad (19)$$

where  $\partial'_\mu = L_{\mu\nu}\partial_\nu$ . We have the Lorentz covariance of Eq.(16) if equations

$$\beta_\mu T L_{\mu\nu} = T\beta_\nu, \quad PT = TP \quad (20)$$

hold. The infinitesimal Lorentz matrix is given by the

$$L_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad (21)$$

where  $\varepsilon_{\mu\nu}$  are six parameters defining rotations and boosts. The matrix  $T$  at the infinitesimal Lorentz transformations reads

$$T = 1 + \frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}, \quad (22)$$

where  $J_{\mu\nu}$  are generators of the Lorentz group in 20-dimensional space. With the aid of Eq.(21),(22) (using the smallness of parameters  $\varepsilon_{\mu\nu}$ ), we obtain from Eq.(20)

$$\beta_\mu J_{\alpha\nu} - J_{\alpha\nu} \beta_\mu = \delta_{\alpha\mu} \beta_\nu - \delta_{\nu\mu} \beta_\alpha, \quad P J_{\alpha\nu} = J_{\alpha\nu} P. \quad (23)$$

The Lorentz group generators in the 20-dimensional representation space are given by

$$\begin{aligned} J_{\mu\nu} &= \beta_\mu \beta_\nu - \beta_\nu \beta_\mu \\ &= \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{[\lambda\mu],[\lambda\nu]} - \varepsilon^{[\lambda\nu],[\lambda\mu]} \\ &\quad + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}} + \varepsilon^{[\widetilde{\lambda\mu}],[\widetilde{\lambda\nu}]} - \varepsilon^{[\widetilde{\lambda\nu}],[\widetilde{\lambda\mu}]}, \end{aligned} \quad (24)$$

and obeys Eq.(23). Thus, we have proved the Lorentz covariance of first-order wave equation (16). In Appendix A, we generalize equations considered on the case of field equations with the source. It is easy to verify with the help of Eq.(12) that the generators (24) obey the usual commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha} J_{\mu\beta} + \delta_{\mu\beta} J_{\nu\alpha} - \delta_{\nu\beta} J_{\mu\alpha} - \delta_{\mu\alpha} J_{\nu\beta}. \quad (25)$$

The Hermitianizing matrix  $\eta$  should satisfy the relations [24]

$$\eta \beta_m = -\beta_m^+ \eta^+, \quad \eta \beta_4 = \beta_4^+ \eta^+ \quad (m = 1, 2, 3). \quad (26)$$

We find

$$\begin{aligned} \eta &= \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2} \varepsilon^{[mn],[mn]} \\ &\quad + \varepsilon^{\widetilde{m},\widetilde{m}} - \varepsilon^{\widetilde{4},\widetilde{4}} + \varepsilon^{[\widetilde{m4}],[\widetilde{m4}]} - \frac{1}{2} \varepsilon^{[\widetilde{mn}],[\widetilde{mn}]}. \end{aligned} \quad (27)$$

The matrix  $\eta$  is the Hermitian matrix,  $\eta^+ = \eta$  and commutes with the projection operator  $P$ :

$$\eta P = P \eta. \quad (28)$$

Consider the “conjugated” wave function

$$\overline{\Psi}(x) = \Psi^+(x) \eta = (\psi_\mu, -\psi_{\mu\nu}, \widetilde{\psi}_\mu, -\widetilde{\psi}_{\mu\nu}), \quad (29)$$

and  $\Psi^+(x)$  is the Hermitian conjugated wave function. We took into account that for neutral fields,  $(\psi_m, \psi_0)$  are real variables. Thus, the relativistically



invariant bilinear form is  $\bar{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x)$ . Then, we obtain from Eq.(16) the “conjugated” equation

$$\bar{\Psi}(x) \left( \beta_\mu \overleftarrow{\partial}_\mu - mP^+ \right) = 0. \quad (30)$$

Formally, one can construct the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \left[ \bar{\Psi}(x) (\beta_\mu \partial_\mu + mP) \Psi(x) - \bar{\Psi}(x) \left( \beta_\mu \overleftarrow{\partial}_\mu - mP^+ \right) \Psi(x) \right]. \quad (31)$$

By varying the action  $S = \int d^4x \mathcal{L}$ , corresponding to the Lagrangian (31), we obtain equations of motion (16), (30). One can check using Eq.(26),(29) that the Lagrangian  $\mathcal{L}$  is the real function,  $\mathcal{L}^* = \mathcal{L}$ . In addition, for neutral fields the equality

$$\bar{\Psi}(x)P^+\Psi(x) = \bar{\Psi}(x)P\Psi(x). \quad (32)$$

is valid although  $P^+ \neq P$ . If one wants to consider charged fields (not photon fields), then the electric current density is given by

$$J_\mu(x) = \frac{i}{m^3} \left[ \left( \partial_\rho F_{\rho\nu}^* \right) \partial_\alpha^2 F_{\nu\mu} - \left( \partial_\alpha^2 F_{\nu\mu}^* \right) \left( \partial_\rho F_{\rho\nu} \right) \right], \quad (33)$$

where the complex conjugation  $*$  does not act on the metric imaginary unit. Using equations of motion (2), one can verify that electric current is conserved  $\partial_\mu J_\mu(x) = 0$ . The electric current density (33) can be cast into the matrix form

$$J_\mu(x) = i\bar{\Psi}(x)\tilde{\beta}_\mu^{(1)}\Psi(x). \quad (34)$$

It follows from Eq.(33) that for the neutral (photon) fields the electric current density vanishes,  $J_\mu(x) = 0$ , as it should be.

## 4 The mass and spin projection operators

Let us consider solutions to Eq.(16) with definite energy and momentum. In the momentum space Eq.(16) becomes

$$\Lambda\Psi(p) = 0, \quad \Lambda = i\hat{p} + mP, \quad \hat{p} = \beta_\mu p_\mu, \quad (35)$$

where  $p_\mu$  is a four-momentum  $p_\mu = (\mathbf{p}, ip_0)$ . Let us consider the massive state,  $p^2 = -m^2$ . For this case the 20-dimensional matrix  $\Lambda$  obeys the equation (see (B5) in Appendix B)

$$\Lambda (\Lambda - m) (\Lambda - 2m) \left( \Lambda^2 - m\Lambda - m^2 \right) = 0. \quad (36)$$

From Eq.(36), we find the solution to Eq.(35) in the form of the matrix

$$\Pi = N (\Lambda - m) (\Lambda - 2m) (\Lambda^2 - m\Lambda - m^2), \quad (37)$$

where  $N$  is a normalization constant, so that  $\Lambda\Pi = 0$ . This means the every column of the matrix  $\Pi$  is the solution to Eq.(35). The requirement that the  $\Pi$  is the projection operator,  $\Pi^2 = \Pi$ , leads to the normalization constant  $N = -1/(2m^4)$  [21]. The projection operator (37) extracts solutions to Eq.(35) for definite energy and momentum corresponding to the massive state.

With the help of the Lorentz group generators (24), we obtain the spin operator (see [21]):

$$\sigma_p = -\frac{i}{2|\mathbf{p}|} \epsilon_{abc} p_a J_{bc} = -\frac{i}{|\mathbf{p}|} \epsilon_{abc} p_a \beta_b \beta_c. \quad (38)$$

The operator (38) obeys the “minimal” matrix equation:

$$\sigma_p (\sigma_p - 1) (\sigma_p + 1) = 0. \quad (39)$$

In accordance with the general method [21], we obtain the projection operators extracting spin projections  $\pm 1$  and 0:

$$S_{(\pm 1)} = \frac{1}{2} \sigma_p (\sigma_p \pm 1), \quad S_{(0)} = 1 - \sigma_p^2, \quad (40)$$

satisfying the relations:  $S_{(\pm 1)}^2 = S_{(\pm 1)}$ ,  $S_{(\pm 1)} S_{(0)} = 0$ ,  $S_{(0)}^2 = S_{(0)}$ .

One may check with the help of Eq.(12) that the operators (40) commute with the mass projection operator (37). As a result, from Eq.(37),(40), we find projection operators

$$\Delta_{\pm 1} = \Pi S_{(\pm 1)}, \quad \Delta_0 = \Pi S_{(0)} \quad (41)$$

extracting solutions to Eq.(35) for definite energy-momentum, spin projections  $\pm 1$ , 0, for states of particles with the mass  $m$ . Eq.(41) defines also the density matrix for pure spin states. It follows from “minimal” polynomial equation (B4) that for the massless state,  $p^2 = 0$ , zero eigenvalues of the matrix  $\Lambda$  are degenerated, and therefore it is impossible to construct solutions to Eq.(35) in the form of the projection operator [21].

## 5 Quantum mechanical Hamiltonian

Now we obtain the quantum mechanical Hamiltonian from equations (6)-(9). The Schrodinger form of equations has some attractive features because non-dynamical components of the wave function are absent. To find the Schrodinger form of Eq.(6)-(9), we exclude the non-dynamical components. Eq.(6)-(9) can be cast in the form of two systems

$$m\psi_{4m} = \partial_4\psi_m - \partial_m\psi_4, \quad m\tilde{\psi}_{4m} = \partial_4\tilde{\psi}_m - \partial_m\tilde{\psi}_4, \quad (42)$$

$$\begin{aligned} \partial_4\psi_{m4} + \partial_n\psi_{mn} &= -m\tilde{\psi}_m, & \partial_4\tilde{\psi}_{m4} + \partial_n\tilde{\psi}_{mn} &= -m\tilde{\psi}_m, \\ m\psi_{mn} = \partial_m\psi_n - \partial_n\psi_m, & m\tilde{\psi}_{mn} = \partial_m\tilde{\psi}_n - \partial_n\tilde{\psi}_m, & m\tilde{\psi}_4 &= \partial_m\tilde{\psi}_{m4}. \end{aligned} \quad (43)$$

We can to exclude auxiliary (non-dynamical) components  $\psi_{mn}$ ,  $\tilde{\psi}_{mn}$ ,  $\tilde{\psi}_4$  from Eq.(43). However, the  $\psi_4$  can not be excluded from Eq.(42). To introduce the evolution of the  $\psi_4$  in time, we use the Lorentz condition  $\partial_m\psi_m + \partial_4\psi_4 = 0$ . After the replacing the non-dynamical components  $\psi_{mn}$ ,  $\tilde{\psi}_{mn}$ ,  $\tilde{\psi}_4$  from Eq.(43) into Eq.(42), we obtain the equations as follows:

$$\begin{aligned} i\partial_t\psi_m &= m\psi_{m4} - \partial_m\psi_4, \\ i\partial_t\psi_4 &= \partial_n\psi_n, \\ i\partial_t\tilde{\psi}_m &= m\tilde{\psi}_{m4} - \partial_m\tilde{\psi}_4, \\ i\partial_t\psi_{n4} &= m\tilde{\psi}_n + \frac{1}{m}(\partial_m\partial_n\psi_m - \partial_m^2\psi_n), \\ i\partial_t\tilde{\psi}_{n4} &= m\tilde{\psi}_n + \frac{1}{m}(\partial_m\partial_n\tilde{\psi}_m - \partial_m^2\tilde{\psi}_n). \end{aligned} \quad (44)$$

Eq.(44) show that 13-components of the wave function  $\Psi(x)$  possess the evolution in time. Therefore, we introduce the 13-component wave function

$$\Phi(x) = \begin{pmatrix} \psi_\mu(x) \\ \psi_{m4}(x) \\ \tilde{\psi}_m(x) \\ \tilde{\psi}_{m4}(x) \end{pmatrix}. \quad (45)$$

With the help of the elements of the matrix algebra Eq.(12), we rewrite Eq.(44) in the Schrodinger form

$$i\partial_t\Phi(x) = \mathcal{H}\Phi(x), \quad (46)$$

where the Hamiltonian is given by

$$\begin{aligned} \mathcal{H} = & m \left( \varepsilon^{n,[n4]} + \varepsilon^{\widetilde{n},[\widetilde{n}4]} + \varepsilon^{[n4],\widetilde{n}} + \varepsilon^{[\widetilde{n}4],\widetilde{n}} \right) + \left( \varepsilon^{4,m} - \varepsilon^{m,4} \right) \partial_m \\ & + \frac{1}{m} \left[ \left( \varepsilon^{[m4],n} + \varepsilon^{[\widetilde{m}4],\widetilde{n}} - \varepsilon^{\widetilde{n},[\widetilde{m}4]} \right) \partial_m \partial_n - \left( \varepsilon^{[m4],m} + \varepsilon^{[\widetilde{m}4],\widetilde{m}} \right) \partial_n^2 \right]. \end{aligned} \quad (47)$$

From the minimal equation (C6), obtained in Appendix C, we find the projection operators extracting states with positive and negative energies for the massless states ( $p^2 = 0$ )

$$\Sigma_{\pm}^0 = \pm \frac{(\mathcal{H} \pm |\mathbf{p}|) \mathcal{H}^2 (\mathcal{H}^2 - \mathbf{p}^2 - m^2) (\mathcal{H}^2 - 2\mathbf{p}^2 - m^2)}{2|\mathbf{p}|^3 m^2 (\mathbf{p}^2 + m^2)}, \quad (48)$$

and massive states ( $p^2 = -m^2$ )

$$\Sigma_{\pm} = \mp \frac{(\mathcal{H} \pm p_0) \mathcal{H}^2 (\mathcal{H}^2 - \mathbf{p}^2) (\mathcal{H}^2 - \mathbf{p}^2 - p_0^2)}{2p_0^3 m^2 \mathbf{p}^2}. \quad (49)$$

Projection operators (48),(49) obey equations as follow:

$$\begin{aligned} \left( \Sigma_{\pm}^0 \right)^2 &= \Sigma_{\pm}^0, \quad \mathcal{H} \Sigma_{\pm}^0 = \pm p_0 \Sigma_{\pm}^0 \quad (p_0 = |\mathbf{p}|), \\ \left( \Sigma_{\pm} \right)^2 &= \Sigma_{\pm}, \quad \mathcal{H} \Sigma_{\pm} = \pm p_0 \Sigma_{\pm} \quad \left( p_0 = \sqrt{|\mathbf{p}|^2 + m^2} \right). \end{aligned} \quad (50)$$

Projection operators (48),(49) can be used to construct physical states in 13-dimensional space of wave functions (45).

## 6 Conclusion

We have formulated Podolsky's generalized electrodynamics equation with higher derivatives in the form of the 20-component first-order relativistic wave equation. This equation describes vector particles possessing the physical massless state and the massive state that is a ghost. To have the consistent theory the mass of the vector state should be very large. One can speculate that the massive vector particles can be described in the gauge-invariant manner by this theory. To have the massive state to be the physical state, we have to use the reverse-sign in the Lagrangian. Then the Hamiltonian also changes the sign. In this case, however, the massless state becomes the ghost and the question arises: how to get rid of it? Therefore the

description of massive particles by Podolsky's generalized electrodynamics is questionable. The relativistically invariant bilinear form, and the Lagrangian were obtained, and these allow us to use the advantages of the formulation of relativistic wave equations. The density matrix obtained can be used for quantum electrodynamics calculations in the first-order formalism. It should be noted that the Petiau-Duffin-Kemmer form of equations was used in quantum chromodynamics [25], i.e. in non-Abelian theory.

The  $13 \times 13$ -matrix Schrodinger form of the equation is derived, and the Hamiltonian is obtained. We found projection operators extracting the physical eigenvalues of the Hamiltonian. The Schrodinger picture has some advantages by considering field interactions.

## Appendix A

Let us consider the field equation (2) with the source of electromagnetic fields – the charge current density:

$$(\partial_\alpha^2 - m^2) \partial_\mu F_{\nu\mu} = -m^2 \tilde{j}_\nu. \quad (A1)$$

We have introduced the current  $\tilde{j}_\nu$  with the same dimension as in classical electrodynamics. The first-order equations (6),(7),(9) remain the same but Eq.(8) is replaced by

$$\partial_\mu \tilde{\psi}_{\nu\mu} + m \tilde{\psi}_\nu = \tilde{j}_\nu(x). \quad (A2)$$

Then Eq.(16) becomes

$$(\beta_\mu \partial_\mu + mP) \Psi(x) = P_0 j(x), \quad (A3)$$

where

$$P_0 = \varepsilon^{\tilde{\mu}, \tilde{\mu}}, \quad j(x) = \begin{pmatrix} j_\mu(x) \\ j_{\mu\nu}(x) \\ \tilde{j}_\mu(x) \\ \tilde{j}_{\mu\nu}(x) \end{pmatrix}, \quad (A4)$$

and  $P_0$  is the projection operator,  $P_0^2 = P_0$ ,  $P_0^+ = P_0$ . The projection operator  $P_0$  extracts only the current  $\tilde{j}_\mu$ . Therefore, the currents  $j_\mu(x)$ ,  $j_{\mu\nu}(x)$ ,  $\tilde{j}_{\mu\nu}(x)$  do not present in the theory and can be put zero. At the Lorentz

transformations,  $j'(x) = Tj(x)$ , and the Lorentz covariance of Eq.(A3) follows from Eq.(20),(23) and

$$P_0 T = T P_0, \quad P_0 J_{\mu\nu} = J_{\mu\nu} P_0. \quad (A5)$$

The Hermitianizing matrix  $\eta$  (27) commutes with  $P_0$ ,  $\eta P_0 = P_0 \eta$ . Then Eq.(30) is replaced by

$$\bar{\Psi}(x) \left( \beta_\mu \overleftarrow{\partial}_\mu - m P^+ \right) = \bar{j}(x) P_0, \quad (A6)$$

where  $\bar{j}(x) = (j_\mu(x), -j_{\mu\nu}(x), \tilde{j}_\mu(x), -\tilde{j}_{\mu\nu}(x))$ . We obtain the classical limit at  $m \rightarrow \infty$  ( $a \rightarrow 0$ ) for Maxwellian electrodynamics from Eq.(A1):

$$\partial_\mu F_{\nu\mu} = \tilde{j}_\nu. \quad (A7)$$

Thus, Eq.(A7) is the standard Maxwell equation with the source term.

## Appendix B

With the help of Eq.(12), we obtain products of matrices entering Eq.(35):

$$\hat{p}^3 = p^2 \hat{p}, \quad \hat{p} P + P \hat{p} = P \hat{p} P + \hat{p}, \quad \hat{p}^2 P = P \hat{p}^2, \quad (B1)$$

$$\hat{p} P \hat{p}^2 = p^2 \hat{p} P, \quad \hat{p} P \hat{p} (1 - P) = \hat{p}^2 (1 - P), \quad (1 - P) \hat{p}^2 P = 0. \quad (B2)$$

Using Eq.(14), (B1), (B2), one finds

$$\Lambda (\Lambda - m) = i m P \hat{p} P - \hat{p}^2,$$

$$\Lambda (\Lambda - m) \left[ \Lambda (\Lambda - m)^2 + 2p^2 (\Lambda - m) + m p^2 \right] = -i p^4 \hat{p} - m p^2 \hat{p}^2 P, \quad (B3)$$

$$\Lambda (\Lambda - m) \left[ \Lambda (\Lambda - m) - m (\Lambda - m) + 2p^2 \right] = i m p^2 \hat{p} - p^2 \hat{p}^2 - m^2 \hat{p}^2 (1 - P).$$

From Eq.(B1)-(B3), we obtain “minimal” polynomials of the matrix  $\Lambda$  for two states:

$$\Lambda^2 (\Lambda - m)^3 = 0, \quad p^2 = 0, \quad (B4)$$

$$\Lambda (\Lambda - m) (\Lambda - 2m) (\Lambda^2 - m \Lambda - m^2) = 0, \quad p^2 = -m^2. \quad (B5)$$

It should be noted that zero eigenvalues of the matrix  $\Lambda$  for the massless state are degenerated.

## Appendix C

From Eq.(47), we obtain the Hamiltonian in the momentum space

$$\begin{aligned} \mathcal{H} = & m \left( \varepsilon^{n,[n4]} + \varepsilon^{\widetilde{n},[\widetilde{n4}]} + \varepsilon^{[n4],\widetilde{n}} + \varepsilon^{[\widetilde{n4}],\widetilde{n}} \right) + ip_m \left( \varepsilon^{4,m} - \varepsilon^{m,4} \right) \\ & + \frac{1}{m} \left[ \left( \varepsilon^{[m4],m} + \varepsilon^{[\widetilde{m4}],\widetilde{m}} \right) \mathbf{p}^2 - \left( \varepsilon^{[m4],n} + \varepsilon^{[\widetilde{m4}],\widetilde{n}} - \varepsilon^{\widetilde{n},[\widetilde{m4}]} \right) p_m p_n \right]. \end{aligned} \quad (C1)$$

Using Eq.(12), one finds

$$\begin{aligned} \mathcal{H}^2 - \mathbf{p}^2 = & m^2 \left( \varepsilon^{[n4],[\widetilde{n4}]} + \varepsilon^{[\widetilde{n4}],[\widetilde{n4}]} + \varepsilon^{\widetilde{n},\widetilde{n}} + \varepsilon^{n,\widetilde{n}} \right) \\ & + imp_n \varepsilon^{4,[n4]} - p_m p_n \left( \varepsilon^{[n4],[m4]} - \varepsilon^{[n4],[\widetilde{m4}]} \right), \end{aligned} \quad (C2)$$

$$\begin{aligned} \mathcal{H}^2 - \mathbf{p}^2 - m^2 = & m^2 \left( \varepsilon^{[n4],[\widetilde{n4}]} - \varepsilon^{[n4],[n4]} + \varepsilon^{n,\widetilde{n}} - \varepsilon^{\mu,\mu} \right) \\ & + imp_n \varepsilon^{4,[n4]} - p_m p_n \left( \varepsilon^{[n4],[m4]} - \varepsilon^{[n4],[\widetilde{m4}]} \right). \end{aligned} \quad (C3)$$

Multiplying Eq.(C2) and Eq.(C3), we obtain

$$\begin{aligned} (\mathcal{H}^2 - \mathbf{p}^2) (\mathcal{H}^2 - \mathbf{p}^2 - m^2) = & im \left( m^2 + \mathbf{p}^2 \right) p_m \left( \varepsilon^{4,[\widetilde{m4}]} - \varepsilon^{4,[m4]} \right) \\ & + \left( m^2 + \mathbf{p}^2 \right) p_m p_n \left( \varepsilon^{[n4],[m4]} - \varepsilon^{[n4],[\widetilde{m4}]} \right). \end{aligned} \quad (C4)$$

Squaring Eq.(C4), one finds

$$\begin{aligned} & \left( \mathcal{H}^2 - \mathbf{p}^2 \right)^2 \left( \mathcal{H}^2 - \mathbf{p}^2 - m^2 \right)^2 \\ = & \mathbf{p}^2 \left( m^2 + \mathbf{p}^2 \right) \left( \mathcal{H}^2 - \mathbf{p}^2 \right) \left( \mathcal{H}^2 - \mathbf{p}^2 - m^2 \right). \end{aligned} \quad (C5)$$

From Eq.(C5), we obtain the “minimal” polynomial of the Hamiltonian

$$\mathcal{H}^2 \left( \mathcal{H}^2 - \mathbf{p}^2 \right) \left( \mathcal{H}^2 - \mathbf{p}^2 - m^2 \right) \left( \mathcal{H}^2 - 2\mathbf{p}^2 - m^2 \right) = 0. \quad (C6)$$

Eigenvalues of the Hamiltonian squared read from Eq.(C6):  $p_0^2 = 0$ ,  $p_0^2 = \mathbf{p}^2$ ,  $p_0^2 = \mathbf{p}^2 + m^2$ ,  $p_0^2 = 2\mathbf{p}^2 + m^2$ . Thus, there are two physical eigenvalues,  $p_0^2 = \mathbf{p}^2$ ,  $p_0^2 = \mathbf{p}^2 + m^2$ , corresponding to massless and massive states of the field, and two nonphysical eigenvalues. Eq.(C6) can be used to find projection operators extracting physical states in the Schrodinger picture.

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